ON THE CONVERGENCE OF FARIMA SEQUENCE TO FRACTIONAL GAUSSIAN NOISE

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ABSTRACT. We consider fractional Gussian noise and FARIMA sequence with Gaussian innovations and show that the suitably scaled distributions of the FARIMA sequences converge to fractional Gaussian noise in the sense of finite dimensional distributions. Finally, we figure out ACF function and estimate the self-similarity parameter H of FARIMA$(0, d, 0)$ by using R/S method.

1. Introduction

Many researchers have studied long range dependent process and self-similar processes which appear in many contexts, for example, in the analysis of traffic load in high speed networks([6], [7]). On the other hand, self-similarity, long range dependence and heavy tailed process have been observed in many time series, i.e. signal processing and finance([4], [6]).

Though the various models proposed for capturing the long range dependent nature of network traffic are all either exactly or asymptotically second order self-similar, their effect on network performance can be very different([8], [9], [10]). Various methods for estimating the self-similarity parameter H or the intensity of long-range dependence in a time series has investigated([9], [11]).

In particular, fractional Gaussian noise and FARIMA sequence in packet network traffic has been the focus of much attention ([5]). And, there has been a recent flood of literature and discussion on the tail behavior of queue-length distribution, motivated by potential applications

to the design and control by high-speed telecommunication networks([1], [2], [3]).

In this paper we consider fractional Gussian noise and FARIMA sequence with Gaussian innovations and show that the suitably scaled distributions of the FARIMA sequences converge in sense of finite dimensional distributions. On the other hand, we describe the R/S method to estimate the self-similarity H of FARIMA sequence.

In section 2, we define long range dependence, self-similar process, fractional Brownian motion, fractional Gaussian noise and FARIMA processes with Gaussian innovations. In section 3, we prove the weak convergence of FARIMA sequence to fractional Gaussian noise. In section 4, we figure out autocorrelation function and estimate the self-similarity parameter H of FARIMA$(0, d, 0)$.

2. Definition and preliminary

In this section we first define short range dependence and long range dependence. Let $\tau_X(k)$ be the covariance of stationary stochastic process $X(t)$.

DEFINITION 2.1. A stationary stochastic process $X(t)$ exhibits *short range dependence* if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| < \infty.$$

DEFINITION 2.2. A stationary stochastic process $X(t)$ exhibits *long range dependence* if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty.$$

A standard example of a long range dependent process is fractional Brownian motion, with Hurst parameter $H > \frac{1}{2}$.

DEFINITION 2.3. A continuous process $X(t)$ is *self-similar with self-similarity parameter* $H \geq 0$ if it satisfies the condition:

$$X(t) \stackrel{d}{=} c^{-H} X(ct), \ \ \forall \, t \geq 0, \forall c > 0,$$

where the equality is in the sense of finite-dimensional distributions.

Self-similar processes are invariant in distribution under scaling of time and space. Brownian motion is a Gaussian process with mean zero and autocovariance function

$$E[X(t_1)X(t_2)] = \min(t_1, t_2).$$

It is H self-similar with $H = 1/2$. And Fractional Brownian motion is important example of self-similar process.

DEFINITION 2.4. A stochastic process $\{B_H(t)\}$ is said to be a *fractional Brownian motion(FBM) with Hurst parameter H* if
  1. $B_H(t)$ has stationary increments
  3. $B_H(0) = 0$ *a.s.*
  4. The increments of $B_H(t)$, $Z(j) = B_H(j+1) - B_H(j)$ satisfy

$$\rho_Z(k) = \frac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2k^{2H}\}$$

DEFINITION 2.5. Let

$$G_j = B_H(j+1) - B_H(j), \ \ j = \cdots, -1, 0, 1, \cdots.$$

The sequence $\{G_j, j \in Z\}$ is called *fractional Gaussian noise* (FGN).

Since fractional Brownian motion $\{B_H(t) : t \in R\}$ has stationary increments, its increments G_j form a stationary sequence. Fractional Gaussian noise is a mean zero and stationary Gaussian time series whose autocovariance function $\tau(h) = EG_iG_{i+h}$ is given by

$$\tau(h) = 2^{-1}\{(h+1)^{2H} - 2h^{2H} + |h-1|^{2H}\},$$

$h \geq 0$. As $h \to \infty$,

$$\tau(h) \sim H(2H-1)h^{2H-2}.$$

Since $\tau(h) = 0$ for $h \geq 1$ when $H = 1/2$. the G_i are white noise. When $1/2 < H < 1$, they display long-range dependence.

We introduce a FARIMA(p, d, q) which is both long range dependent and has heavy tails. FARIMA(p, d, q) processes are capable of modeling both short and long range dependence in traffic models since the effect of d on distant samples decays hyperbolically while the effects of p and q decay exponentially.

DEFINITION 2.6. A stationary process X_t is called a *FARIMA(p, d, q) process* if

$$\phi(B)\nabla^d X_t = \theta(B)Z_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and the coefficients $\phi_1, \cdots, \phi_p$ and $\theta_1, \cdots, \theta_q$ are constants,

$$\nabla^d = (1 - B)^d = \sum_{i=0}^{\infty} b_i(-d)B^i$$

and B is the backward shift operator defined as $B^i X_t = X_{t-i}$ and

$$b_i(-d) = \prod_{k=1}^{i} \frac{k + d - 1}{k} = \frac{\Gamma(i + d)}{\Gamma(d)\Gamma(i + 1)}.$$

For large lags d, the autocovariance function satisfies for $0 < d < 1/2$,

$$\tau(h) \sim C_d h^{2d-1} \quad \text{as} \quad h \to \infty$$

where $C_d = \pi^{-1}\Gamma(1 - 2d)\sin(\pi d)$. Thus, for large lags d, the autocovariance function has the same power decay as the autocovariance of fractional Gaussian noise. Relating the exponents gives

$$d = H - \frac{1}{2}.$$

3. Weak convergence of FARIMA sequence to fractional Gaussian noise

LEMMA 3.1. *Fix* $0 < H < 1$ *and let* $\{Z_j, j = \cdots, -1, 0, 1, \cdots\}$ *be a stationary Gaussian sequence with mean zero and autocovariance function* $\tau(j) = EZ_0 Z_j$ *satisfying:*
(i) Case $1/2 < H < 1$ *:*

$$\tau(j) \sim cj^{2H-2} \quad as \quad j \to \infty \quad with \quad c > 0;$$

(ii) Case $H = 1/2$ *:*

$$\sum_{j=1}^{\infty} |\tau(j)| < \infty, \quad \sum_{-\infty}^{\infty} \tau(j) = c;$$

(iii) Case $0 < H < 1/2$ *:*

$$\tau(j) \sim cj^{2H-2} \quad as \quad j \to \infty \quad with \quad c < 0 \quad and \quad \sum_{-\infty}^{\infty} \tau(j) = c;$$

Then the finite dimensional distributions of $\{N^{-H} \sum_{j=1}^{[Nt]} Z_j, 0 \le t \le 1\}$ *converge to those of* $\{\sigma_0 B_H(t), 0 \le t \le 1\}$ *where*

$$\sigma_0^2 = \begin{cases} H^{-1}(2H - 1)^{-1}c & \text{if} \quad 1/2 < H < 1, \\ c & \text{if} \quad H = 1/2, \\ -H^{-1}(2H - 1)^{-1}c & \text{if} \quad 0 < H < 1/2. \end{cases}$$

Proof. (Theorem 7.2.11 of [5]) □

THEOREM 3.2.

$$\frac{1}{N^H} \frac{1}{M^{1/2}} \sum_{k=iN+1}^{(i+1)N} \sum_{j=0}^{M} b_j(-d)a_{k-j}$$

converges to $\sigma_0 G_i$ in the sense of finite dimensional distributions, as $M \to \infty$ and $T \to \infty$, where, $\sigma_0^2 = \frac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}$.

Proof. By Lemma 2 of [4],

$$\lim_{M\to\infty} \frac{1}{M^{1/2}} \sum_{j=0}^{M} b_j(-d)a_{k-j} = G_H(k).$$

Here, $G_H(k)$ represents a stationary Gaussian process whose covariance function has the form $\tau(k) \sim ck^{2H-2}$ and $1/2 < H < 1$.
And, the covariance function of $\sum_{j=0}^{\infty} b_j(-d)a_{k-j}$,

$$\begin{aligned} \tau(k) &\sim \frac{\Gamma(1-2d)\sin(\pi d)}{\pi} k^{2d-1} \\ &= \frac{-\Gamma(2-2H)\cos(\pi H)}{\pi} k^{2H-2} \end{aligned}$$

where $H = d + 1/2$, has the same form as ck^{2H-2}. Therefore,

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{N^H} \frac{1}{M^{1/2}} \sum_{k=iN+1}^{(i+1)N} \sum_{j=0}^{M} b_j(-d)a_{k-j} = \frac{1}{N^H} \sum_{k=iN+1}^{(i+1)N} G_H(k)$$

converges to $\sigma_0 G_i$ in the sense of finite dimensional distributions.
By Lemma 3.1, with $\sigma_0 = \sqrt{\frac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}}$,

$$N^{-H} \sum_{k=iN+1}^{(i+1)N} G_H(k) = N^{-H} \sum_{k=1}^{(i+1)N} G_H(k) - N^{-H} \sum_{k=1}^{iN} G_H(k)$$

converge to

$$\sigma_0 B_H(i+1) - \sigma_0 B_H(i) = \sigma_0 G_i.$$

□

THEOREM 3.3. *Let X_i be the autoregressive process of order one, i.e. $X_i = \phi_1 X_{i-1} + a_i$, where $a_i \sim N(0,1)$ for each i. Then*

$$\lim_{N\to\infty} \frac{1}{N^H} \sum_{k=iN+1}^{(i+1)N} X_k = \sigma_0 G_i,$$

where, $\sigma_0 = \sqrt{\phi_1/1 - \phi_1}$

Proof. We know that

$$(1 - \phi_1 B)X_i = a_i,$$

i.e.

$$X_i = \sum_{k=0}^{\infty} \phi_1^k a_{i-k}.$$

And, we get

$$Cov_X(k) = \phi_1^k, \ k \geq 1, |\phi_1| < 1.$$

Therefore,

$$\tau(k) = \phi_1^k.$$

Since

$$\sum_{k=1}^{\infty} \tau(k) = \sum_{k=1}^{\infty} \phi_1^k = \frac{\phi_1}{1 - \phi_1} < \infty,$$

by Lemma 3.1 (b), we get

$$\lim_{N \to \infty} \frac{1}{N^H} \sum_{k=iN+1}^{(i+1)N} X_k = \sqrt{\phi_1/(1 - \phi_1)} G_i.$$

$\square$

For each $N \geq 1$, the transformation

$$T_N : Z \to T_N Z = \{(T_N Z)_i, i = \cdots, -1, 0, 1, \cdots\},$$

where

$$(T_N Z)_i = \frac{1}{N^H} \sum_{j=iN+1}^{(i+1)N} Z_j, \ \ i = \cdots, -1, 0, 1, \cdots.$$

Let $\{X_H(t) : t \in R\}$ be a H self-similar with stationary increments. Then its increments

$$Y_j = X_H(j+1) - X_H(j), \ \ j = \cdots, -1, 0, 1, \cdots$$

is a stationary sequence.

THEOREM 3.4.

$$T_N Y \overset{d}{=} Y$$

Proof. For any $\theta_1, \theta_2, \cdots, \theta^d$, $d \geq 1$, and $N \geq 1$,

$$
\begin{aligned}
\sum_{i=1}^{d} \theta_i \frac{1}{N^H} \sum_{j=iN+1}^{(i+1)N} Y_j &= \sum_{i=1}^{d} \theta_i \frac{1}{N^H} (X_H((i+1)N) - X_H(iN)) \\
&\stackrel{d}{=} \sum_{i=1}^{d} \theta_i (X_H((i+1)) - X_H(i)) \\
&= \sum_{i=1}^{d} \theta_i Y_i.
\end{aligned}
$$

$\square$

COROLLARY 3.5. *Fractional Gaussian noise is the only Gaussian sequence satisfying $T_N Y \stackrel{d}{=} Y$.*

Proof. It follows from the fact that fractional Brownian motion is the unique Gaussian H self-similar process with stationary increments. $\square$

4. Estimation of the self-similarity parameter of ARIMA sequence with Gaussian innovations

When $d < 1/2$, the FARIMA process is stationary and the covariance function of a FARIMA$(0, d, 0)$ process with zero mean and unit variance Gaussian innovations has the form

$$
\begin{aligned}
\tau(k) &= \frac{(-1)^k (-2d)!}{(\pi - d)!(-k - d)!} \\
&\sim \frac{\Gamma(1 - 2d)\sin(\pi d)}{\pi} k^{2d-1} \quad as \ k \to \infty
\end{aligned}
$$

The covariance function of the generalized FARIMA(p, d, q) processes with Gaussian innovations has additional short-term components but follows the same asymptotic relation as the covariance function as FARIMA$(0, d, 0)$ processes.

Hence, we consider FARIMA$(0, d, 0)$ in terms of $d = 0.2$ and estimate the self-similarity parameter H.

The R/S method which was used by Taqqu and Willinger([10]) is one of the better known method. For a time series $X = \{X_i : i \leq 1\}$, with partial sum $Y(n) = \sum_{i=1}^{n} X_i$, and sample variance $S^2(n) = $

$(1/n) \sum_{i=1}^{n} X_i^2 - (1/n)^2 Y(n)^2$, the R/S static, or the rescaled adjusted range, is given by

$$\frac{R}{S}(n) = \frac{1}{S(n)} \left[\max_{0 \leq t \leq n}(Y(t) - \frac{t}{n}Y(n)) - \min_{0 \leq t \leq n}(Y(t) - \frac{t}{n}Y(n)) \right].$$

For fractional Gaussian noise

$$E[R/S(n)] \sim C_H n^H, \quad \text{as} \ \ n \rightarrow \infty,$$

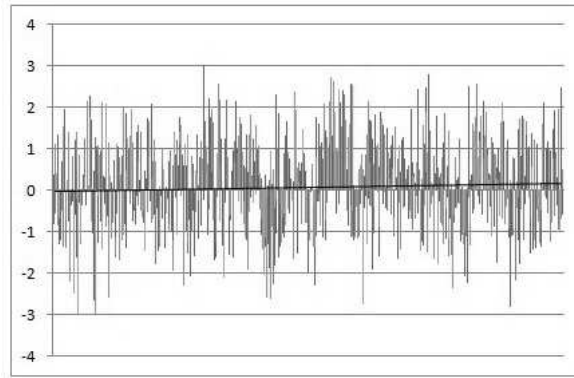where C_H is positive, finite constant not depend on n.

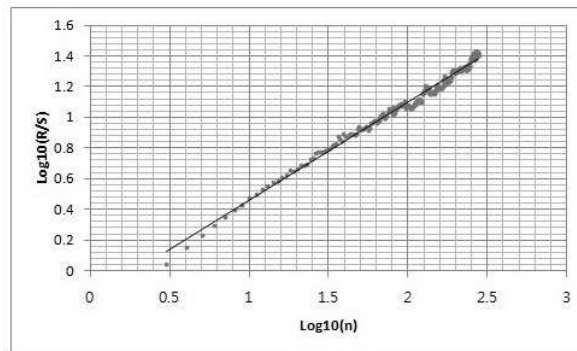FIGURE 1. Simulated FARIMA$(0, 0.2, 0)$, $n = 1,000$

FIGURE 2. Estimating H

The following figure 1 and figure 3 illustrate FARIMA processes in the case $n = 1,000$ and $n = 10,000$.

To determine H using the R/S statistic, proceed as follows. For a time series of length N, subdivide the series into blocks. Then, for each lag n, compute $R(n)/S(n)$. Choosing logarithmically spaced values of n, plot $log[R(n)/S(n)]$ versus $\log(n)$ and get, for each n, several points on the plot.

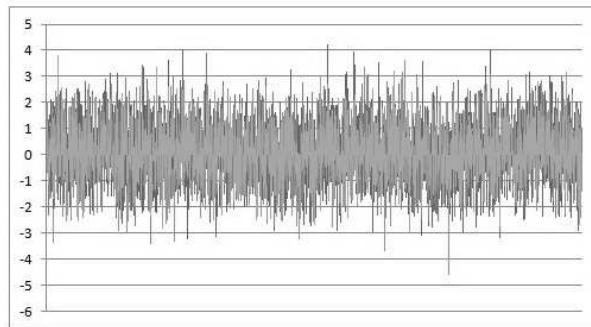In figure 2 and figure 4, we estimate H as 0.6859 and 0.6968.
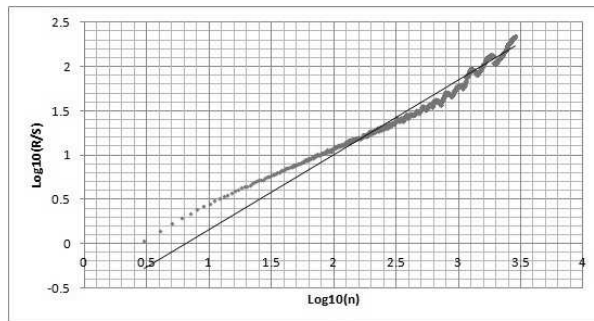

FIGURE 3. Simulated FARIMA$(0, 0.2, 0)$, $n = 10,000$

FIGURE 4. Estimating H

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